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SENSOR FAULT DIAGNOSIS BASED ON A $\mathcal{H}_\infty$ SLIDING MODE AND UNKNOWN INPUT OBSERVER FOR TAKAGI-SUGENO SYSTEMS WITH UNCERTAIN PREMISE VARIABLES

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ABSTRACT

This paper presents the design of a $\mathcal{H}_\infty$ sliding mode and an unknown input observer for Takagi-Sugeno (TS) systems. Contrary to the common approaches reported in the literature, which considers exact premise variables, this work deals with the problem of inexact measurements of the premise variables. The proposed method is based on a $\mathcal{H}_\infty$ criteria to be robust to disturbances, sensor noise and uncertainty on the premise variables. The observer convergence and stability are established by considering a quadratic Lyapunov function, which relies on a set of Linear Matrix Inequalities. Then, a dedicated observer scheme is considered to detect and isolate sensor faults. Finally, the performance and applicability of the proposed approach are illustrated through numerical experiments on a nonlinear model that represents the lateral dynamics of an electric vehicle.

Key Words: Takagi-Sugeno system, fault diagnosis, uncertain premise variables, sliding mode observer.

I. INTRODUCTION

In the last years, the study of fault detection and isolation (FDI) methods has increased substantially, this is due to the fact of the increasing industrial demand of performance, safety and reliability [1,2]. Among the different approaches to design FDI systems, the FDI based on state observers has proved to be one of the best trade-offs between performance and applicability [3]. A wide range of FDI methods based on observers can be found in the literature, for instance, by considering a Kalman filter [4], $\mathcal{H}_\infty$ filter [5–7], FDI and fault tolerant control with time-delay [5,8], mixed $\mathcal{H}_\infty/\mathcal{H}_-$ [9,10], unknown input observers [11], sliding mode [12,13], fault detection and fault estimation based on descriptor systems [14–16], among others. It is also clear that a reliable FDI system requires realistic models, based on data or first principles, that represent the complex dynamics of real physical systems. These dynamics commonly have a nonlinear behavior, and therefore the mathematical models are also nonlinear. Nonlinear models are complex to analyze and consequently, designing nonlinear FDI schemes for this kind of systems is still a difficult task from the theoretical and practical point of view. However, it has been proved that FDI schemes that consider both non-linearities and non-stationary exogenous parameters are less conservative and easy to design than nonlinear approaches [17]. Recently, the Takagi-Sugeno (TS) systems representation has proved to be an interesting approach to deal with complex nonlinear systems [18].

TS models describe nonlinear systems through a collection of time-invariant linear models (LTI) interpolated by nonlinear functions known as weighting functions. These local models can be obtained from the well-known, nonlinear sector approach, which transforms the nonlinear system into a polytopic TS system [19]. Note that there exists an equivalence between TS and linear parameter varying systems (LPV), due to the fact that, LPV and TS models are both obtained from the nonlinear-sector transformation, which is commonly called as a quasi-LPV (qLPV) system [20]. Then, the scheduling variables mentioned in qLPV systems are

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analogous to the term premise variables in TS systems [3,21]. In the literature, the premise variables are classified into two types: (i) the measurable premise variables that depend mainly on the inputs or outputs of the system and on the non-stationary exogenous parameters [18]; and (ii) the unmeasurable premise variables that depends on unmeasurable states, *i.e.* when there is not a sensor to measure a premise parameter or when the cost of the sensors is prohibitive [22].

The design of fault diagnosis systems based on TS observers has been receiving considerable attention as can be consulted in the recent book [3]. Nonetheless, most of the works consider the case of measurable premise variables, moreover, it is considered that such variable is measured with high precision. However, from a practical point of view, these premise variables are measured with a certain degree of uncertainty, *e.g.* sensors with offsets, low resolutions, imprecisions due to calibration, weather changes, instrument quality, *etc.* [23]. In this case, it is necessary to consider inexact premise variables in order to design a reliable and effective FDI system. Few results have been developed considering the uncertain case. For instance, in [24] the design of a LPV controller was presented considering inaccurate scheduling variables. In [25] a filter, which considers additive uncertainty in the scheduling variables, was formulated. This work was extended with additive-multiplicative uncertainty in [23]. In [17] an uncertain sliding mode LPV observer was proposed and applied to the lateral dynamics of an electric vehicle. More recently, in [26], an FDI system based on a sliding mode observer was considered to estimate actuators faults. However, to the best knowledge of the authors, the fault detection approach, which considers time-varying outputs and unknown inputs has not been well investigated yet [27].

In this paper, the design of a $\mathcal{H}_\infty$ sliding mode and unknown input TS observer is presented. The proposed approach considers inexact premise variables and time-variant output and its extension to the detection and isolation of faults in sensors. The performance criteria is satisfied to be robust to disturbances, sensor noise and measurement uncertainty on the premise variables by solving a set of linear matrix inequalities (LMIs), which are obtained through a Lyapunov formalism. Unlike the results in [17,26], the proposed method considers: (i) a TS system with uncertain premise variables presented in Section II; (ii) a time-varying output in the development of the $\mathcal{H}_\infty$ sliding mode and unknown input observer presented in Section III; (iii) relaxed LMI conditions, which increase the applicability of the method; and (iv) a dedicated observer scheme for the FDI and its application on the lateral dynamics of an electric vehicle presented in Section IV. Conclusions are given in Section VI.

II. PROBLEM STATEMENT

Consider a Takagi-Sugeno system described by:

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{q} \mu_i(\xi(t)) \left(A_i x(t) + B_{1i} u(t) + B_{2i} \omega(t) \right), \\
y &= \sum_{i=1}^{q} \mu_i(\xi(t)) C_i x(t) + D v(t),
\end{aligned} \tag{1}
$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $\omega(t)$, $y(t) \in \mathbb{R}^p$, and $v(t)$ are the state vector, input vector, disturbance, output vector and sensor noise respectively. A_i, B_{1i}, B_{2i}, C_i, and D are known matrices with compatible dimensions. q represents the number of linear sub-models and $\mu_i \in \mathbb{R}$ are weighting functions that depend on $\xi(t)$, which are also known as premise variables or scheduling variables [28]. The weighting functions satisfy the convex sum:

$$
\begin{aligned}
&\forall i \in \left[1, 2, \ldots, q\right], \ \mu_i(\xi(t)) \geq 0, \\
&\sum_{i=1}^{q} \mu_i(\xi(t)) = 1, \ \forall t.
\end{aligned} \tag{2}
$$

Commonly, the premise variables depend on non-stationary exogenous parameters measured online or measured states. Several works have been published over the last years assuming that the premise variables are perfectly measurable as described in [18]. Nevertheless, in many practical applications, the premise variables are measured with certain levels of uncertainty or are not measurable [23,26,29]. For example, in a TS model of an electric vehicle, the premise variable is given by the longitudinal speed, which is assumed to be measured with precision [30]. However, in practice, the longitudinal speed is estimated from the number of turns of the tires with respect to time, whose measurements are affected by the surface of the road, skidding, sensor noise, among others. In other words, the longitudinal speed is measured inexactly with a certain degree of uncertainty [17]. In this case, the traditional methods are not applicable and it is necessary to consider a strategy to deal with the uncertainty in those measures. This problem is addressed in this work.

Before presenting the main results, some assumptions are considered:

- Matrices A_i are Hurwitz.
- The TS system (1) is locally observable.
- The disturbance vector $\omega(t)$ is unknown and bounded:

$$\| \omega(t) \| < \delta_1, \tag{3}$$

where δ_1 is a known positive scalar.

Note that, in the case of perfectly measured premise variables, the weighting functions can be used directly in the design of the control algorithms such as state observers or controllers. Nonetheless, in the case of premise variables with uncertainties, there exist mismatches between the weighting functions that can deteriorate or destabilize the observer or controller. In this case, it is necessary to use a robust approach that considers uncertain weighting functions in order to guarantee the stability and performance of the proposed algorithm. In this work, the weighting functions are considered as:

$$\mu_i(\xi(t)) = \rho_i(t) * \hat{\mu}_i(\hat{\xi}(t)), \tag{4}$$

where $\hat{\mu}_i(\hat{\xi}(t))$ are the uncertain weighting functions due to an inaccurate measure of the premise variables $\hat{\xi}(t)$; $\rho_i(t)$ is the uncertain factor, whose minimum and maximum values are given by $\underline{\rho}_i$ and $\overline{\rho}_i$, respectively. Then, by considering (4), system (1) can be rewritten as an uncertain system as:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \left(\bar{A}_i x(t) + \bar{B}_{1i} u(t) + \bar{B}_{2i}\omega(t) \right), \\ y &= \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \bar{C}_i x(t) + Dv(t), \end{aligned} \tag{5}$$

where:

$$\begin{aligned} \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \bar{A}_i &= \sum_{i=1}^{q} \rho_i(t) \hat{\mu}_i(\hat{\xi}(t)) A_i \\ &= \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \left(A_i + (\rho_i(t) - 1) A_i \right) \\ &= \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \left(A_i + \Delta A_i(t) \right), \end{aligned} \tag{6}$$

$$\begin{aligned} \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \bar{B}_{1i} &= \sum_{i=1}^{q} \rho_i(t) \hat{\mu}_i(\hat{\xi}(t)) B_{1i} \\ &= \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \left(B_{1i} + (\rho_i(t) - 1) B_{1i} \right) \\ &= \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \left(B_{1i} + \Delta B_{1i}(t) \right), \end{aligned} \tag{7}$$

$$\begin{aligned} \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \bar{B}_{2i}\omega(t) &= \sum_{i=1}^{q} \rho_i(t) \hat{\mu}_i(\hat{\xi}(t)) B_{2i}\omega(t) \\ &= \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) B_{2i}\omega_1(t), \end{aligned} \tag{8}$$

and,

$$\begin{aligned} \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \bar{C}_i &= \sum_{i=1}^{q} \rho_i(t) \hat{\mu}_i(\hat{\xi}(t)) C_i \\ &= \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \left(C_i + (\rho_i(t) - 1) C_i \right) \\ &= \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \left(C_i + \Delta C_i(t) \right), \end{aligned} \tag{9}$$

with:

$$\begin{aligned} \Delta A_i(t) &= (\rho_i(t) - 1) A_i, \\ \Delta B_i(t) &= (\rho_i(t) - 1) B_i, \\ \Delta C_i(t) &= (\rho_i(t) - 1) C_i, \\ \omega_1(t) &= \rho_i(t)\omega(t). \end{aligned}$$

In order to facilitate the observer design, the uncertainties and the external noise are bounded as follows:

$$\|\Delta A_i(t)\| \leq \zeta_{1i}, \tag{10}$$

$$\|\Delta C_i(t)\| \leq \zeta_{2i}, \tag{11}$$

$$\|\omega_1(t)\| \leq \delta_2, \tag{12}$$

with positives scalars ζ_{1i}, ζ_{2i} and δ_2. Since ρ_i has a lower and a upper bound, the matrix $\Delta B_{1i}(t)$ can be rewritten as:

$$\Delta B_{1i}(t) = \delta_{3i} M B_{1i}, \tag{13}$$

where δ_{3i} is a positive scalar with the value of $\delta_{3i} = \max\{| \overline{\rho}_i - 1 |, | \underline{\rho}_i - 1 |\}$ and $\|M\| \leq 1$.

Remark 1. In most of the works, the output matrix is considered as constant. Nevertheless, in some applications this matrix is not constant, *e.g.* [30,31]. In this case, the uncertainty also affects matrices C_i, but considering a time-varying matrix is not trivial due to the fact that affects all the observer design and increases the difficulty. However, considering a non-constant output matrix is important in order to increase the applicability of the observer.

Under the assumption that the uncertain system (5) is locally observable, the following sliding mode observer is proposed:

$$\hat{x}(t) = \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t))\Big(A_i\hat{x}(t) + B_{1i}u(t) + L_i(y(t) - \hat{y}(t)) + \varphi_i(t) + \sigma_i(t)\Big),$$
$$\hat{y} = \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t))C_i\hat{x}(t), \tag{14}$$

where $\hat{x}(t)$ represents the estimated states, $\varphi_i(t)$ and $\sigma_i(t)$ are discontinuous functions, $\hat{y}(t)$ is the estimated output vector, L_i are unknown gain matrices of appropriate dimensions that has to be computed.

Then, the challenge is to find the L_i matrices such that the estimation error between systems (5) and (14) converge asymptotically to zero despite disturbances, sensor noise and uncertainties. In the following section the design conditions of the Takagi-Sugeno observer are established.

III. MAIN RESULT: OBSERVER DESIGN

In order to establish the conditions for the asymptotic convergence of the observer (14), let us define the estimation error as:

$$e(t) = x(t) - \hat{x}(t). \tag{15}$$

The dynamic of the estimation error can be evaluated using the Equations (6)-(9) and (14), such that:

$$\dot{e}(t) = \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \sum_{j=1}^{q} \hat{\mu}_j(\hat{\xi}(t))\Big((A_i - L_iC_j)e(t) + (\Delta A_i - L_i\Delta C_j)x(t) + B_{3i}v(t) + B_{2i}\omega_1(t) + \varphi_i(t) + \sigma_i(t)\Big), \tag{16}$$

with

$$B_{3i} = \hat{B}_{1i} + \Delta_{3i}M\tilde{B}_{1i}, \tag{17}$$

$$\hat{B}_{1i} = [0 \quad -L_iD], \tag{18}$$

$$\tilde{B}_{1i} = [B_{1i} \ \ 0], \tag{19}$$

$$\Delta_{3i} = \delta_{3i}I, \tag{20}$$

$$v(t) = [u(t) \ \ v(t)]^T. \tag{21}$$

It can be seen from (16) that the gains to be tuned, the state vector x, the unknown external input, $\varphi_i(t)$ and $\sigma_i(t)$ are also involved in the error estimation. Since the main criterion of selecting the gain L_i is to make stable the estimation error system such that the estimation error converge to zero, we define a new variable as:

$$z(t) = Ge(t), \tag{22}$$

with a constant matrix G. Then, the challenges and objectives are to tune the observer gain L_i and discontinuous functions $\varphi_i(t)$ and $\sigma_i(t)$ such that:

- The estimation error system in (16) is asymptotically stable when the external input is zero.
- The effect from the external input $v(t)$ to the signal $z(t)$ is constrained as:

$$\|z(t)\|_2 < \gamma\|v(t)\|_2, \tag{23}$$

  where $\|X\|_2$ denotes the 2-norm of $\mathcal{L}_2$-bounded signal X and γ is the $\mathcal{H}_\infty$ performance index.
- The discontinuous functions $\varphi_i(t)$ and $\sigma_i(t)$ can compensate for the effects of $\omega_1(t)$ and $\Delta A_i - L_i\Delta C_j$ respectively.

Before proceeding, the following lemmas are presented.

Lemma 1 [32]. For matrices X and Y with appropriate dimensions, the following properly holds for any positive scalar ψ:

$$X^TY + Y^TX \le \psi X^TX + \psi^{-1}Y^TY.$$

Lemma 2 [33]. If there exist real matrices $\Xi = \Xi^T$, $\hat{E}$ and $\hat{F}$ with compatible dimensions, and M satisfying $\| M \| \le 1$, then, the following condition:

$$\Xi + \hat{E}M\hat{F} + (\hat{E}M\hat{F})^T < 0, \tag{24}$$

is satisfied if and only if there is a positive scalar $\psi > 0$ such that

$$\begin{bmatrix} \Xi & \hat{E} & \psi\hat{F}^T \\ * & -\psi I & 0 \\ * & * & -\psi I \end{bmatrix} < 0. \tag{25}$$

Lemma 3 [32]. Let Ψ_{ij} be matrices of appropriate dimensions where $i,j \to \{1,\dots,q\}$. Then $\Psi_{ij} < 0$ holds if

$$\Psi_{ii} < 0, \quad \forall i, \tag{26}$$

$$\frac{2}{q-1}\Psi_{ii} + \Psi_{ij} + \Psi_{ji} < 0, \quad i \neq j. \tag{27}$$

The following theorem provides the conditions for the asymptotic stability and the $\mathcal{H}_\infty$ performance of the estimation error in (16).

Theorem 1. Given the TS system (1) and the state observer (14), the estimation error (16) is asymptotically stable with $\mathcal{H}_\infty$ performance and attenuation level $\gamma > 0$, if there exist a matrix $P = P^T > 0$, matrices W_i and positive scalars ψ_1, ψ_2, ψ_3, ψ_4 and ψ_5 such that following optimization problem has solution:

$$\min \quad \gamma$$

under the following LMI constraints:

$$\begin{bmatrix} \Lambda_{ij} & PB_{3i} & P & W_i \\ * & -\gamma^2 I & 0 & 0 \\ * & * & -\psi_1 I & 0 \\ * & * & * & -\psi_3 I \end{bmatrix} < 0, \tag{28}$$

with:

$$\begin{aligned} \Lambda_{ij} &= A_i^T P + PA_i - C_j^T W_i^T - W_i C_j + \psi_2 \zeta_{1i}^2 I \\ &\quad + \psi_4 \zeta_{2i}^2 I + \psi_5 I + G^T G, \\ e_y(t) &= y(t) - \hat{y}(t), \end{aligned}$$

if $e_y(t) \neq 0$, then

$$\begin{aligned} \varphi_i(t) &= \delta_2^2 \psi_5^{-1} \frac{\|PB_{2i}\|^2}{2e_y(t)^T e_y(t)} P^{-1} \sum_{j=1}^{q} \hat{\mu}_j(\hat{\xi}(t)) C_j^T e_y(t), \\ \sigma_i(t) &= \psi_{9i} \frac{\hat{x}(t)^T \hat{x}(t)}{2e_y(t)^T e_y(t)} P^{-1} \sum_{j=1}^{q} \hat{\mu}_j(\hat{\xi}(t)) C_j^T e_y(t). \end{aligned}$$

If $e_y = 0$, then

$$\begin{aligned} \varphi_i(t) &= 0, \\ \sigma_i(t) &= 0, \end{aligned}$$

where:

$$\begin{aligned} \psi_6 &= \frac{\psi_1}{\psi_2 - \psi_1}, \\ \psi_7 &= \frac{\psi_3}{\psi_4 - \psi_3}, \\ \psi_{9,i} &= \psi_1(1+\psi_6)\zeta_{1i}^2 + \psi_3(1+\psi_7)\zeta_{2i}^2. \end{aligned}$$

Then, the observer parameters are computed by:

$$L_i = P^{-1} W_i. \tag{29}$$

Proof. Consider the following performance criteria:

$$\mathcal{J}(t) := \dot{V}(t) + z(t)^T z(t) - \gamma \nu(t)^T \nu(t) < 0, \tag{30}$$

where $V(t)$ is the Lyapunov function which is selected as $V = e(t)^T P e(t)$, with $P = P^T > 0$, such that:

$$\begin{aligned} \mathcal{J}(t) &:= \dot{e}(t)^T P e(t) + e(t)^T P \dot{e}(t) \\ &\quad + z(t)^T z(t) - \gamma^2 \nu(t)^T \nu(t). \end{aligned} \tag{31}$$

Then, by considering (16), the following inequality is obtained:

$$\begin{aligned} \mathcal{J}(t) &:= \sum_{i=1}^{q}\sum_{j=1}^{q} \hat{\mu}_i(\hat{\xi}(t))\hat{\mu}_j(\hat{\xi}(t)) \\ &\quad \times \Big(e(t)^T \left((A_i - L_i C_j)^T P + P(A_i - L_i C_j) \right) e(t) \\ &\quad + x(t)^T \Delta A_i^T P e(t) + e(t)^T P \Delta A_i x(t) \\ &\quad + x(t)^T \Delta C_j^T L_i^T P e(t) + e(t)^T P L_i \Delta C_j x(t) \\ &\quad + \omega_1(t)^T B_{2i}^T P e(t) + e^T P B_{2i} \omega_1(t) \\ &\quad + 2e(t)^T P B_{3i} \nu(t) + 2e(t)^T P \varphi_i(t) + 2e(t)^T P \sigma_i(t) \\ &\quad + e(t)^T G^T G e(t) - \gamma^2 \nu(t)^T \nu(t) \Big) < 0. \end{aligned} \tag{32}$$

In order to get an expression in terms of Lemma 1, Equation (32) can be rewritten as:

$$\begin{aligned} & x(t)^T \Delta A_i^T P e(t) + e(t)^T P \Delta A_i x(t) \\ & \leq \psi_1^{-1} (Pe(t))^T P e(t) + \psi_1 x(t)^T \Delta A_i^T \Delta A_i x(t), \end{aligned} \tag{33}$$

considering (10), the following relationship is established:

$$\begin{aligned} & \psi_1^{-1} (Pe(t))^T P e(t) + \psi_1 x(t)^T \Delta A_i^T \Delta A_i x(t) \\ & \leq \psi_1^{-1} e(t)^T P^2 e(t) + \psi_1 \zeta_{1i}^2 x(t)^T x(t), \end{aligned} \tag{34}$$

where $x(t) = e(t) + \hat{x}(t)$, then, the expression (34) becomes:

$$\begin{aligned} & \psi_1^{-1} e(t)^T P^2 e(t) + \psi_1 \zeta_{1i}^2 \left(e(t) + \hat{x}(t)\right)^T \left(e(t) + \hat{x}(t)\right) \\ & = \psi_1^{-1} e(t)^T P^2 e(t) + \psi_1 \zeta_{1i}^2 \big(e(t)^T e(t) + \hat{x}(t)^T e(t) \\ & \quad + e(t)^T \hat{x}(t) + \hat{x}(t)^T \hat{x}(t) \big). \end{aligned} \tag{35}$$

Using again the Lemma 1, the last expression can be rewritten as follows:

$$\begin{aligned}&\psi_1^{-1}e(t)^TP^2e(t)+\psi_1\zeta_{1i}^2\big(e(t)^Te(t)\\&\quad+\hat{x}(t)^Te(t)+e(t)^T\hat{x}(t)+\hat{x}(t)^T\hat{x}(t)\big)\\&\le\psi_1^{-1}e(t)^TP^2e(t)+\psi_2\zeta_{1i}^2e(t)^Te(t)\\&\quad+\psi_1(1+\psi_6)\zeta_{1i}^2\hat{x}(t)^T\hat{x}(t),\end{aligned}\tag{36}$$

where $\psi_2=\psi_1(1+\psi_6^{-1})$.

Using the previous procedure one gets:

$$\begin{aligned}&x(t)^T\Delta C_j^TL_i^TPe(t)+e(t)^TPL_i\Delta C_jx(t)\\&\le\psi_3^{-1}e(t)^TPL_iL_i^TPe(t)+\psi_3\zeta_{2j}^2x(t)^Tx(t)\\&\le\psi_3^{-1}e(t)^TPL_iL_i^TPe(t)+\psi_3\zeta_{2j}^2(e(t)\\&\quad+\hat{x}(t))^T(e(t)+\hat{x}(t))\\&=\psi_3^{-1}e(t)^TPL_iL_i^TPe(t)+\psi_3\zeta_{2j}^2\big(e(t)^Te(t)\\&\quad+\hat{x}(t)^Te(t)+e(t)^T\hat{x}(t)+\hat{x}(t)^T\hat{x}(t)\big)\\&\le\psi_3^{-1}e(t)^TPL_iL_i^TPe(t)+\psi_4\zeta_{2j}^2e(t)^Te(t)\\&\quad+\psi_3(1+\psi_7)\zeta_{2j}^2\hat{x}(t)^T\hat{x}(t),\end{aligned}\tag{37}$$

where $\psi_4=\psi_3(1+\psi_7^{-1})$.

Two cases can therefore be distinguished according to the value of the output residual:

Case 1. If $e_y=0$, since each subsystem is observable, then, the estimation error is zero.
Case 2. If $e_y\neq 0$, then, the following expression holds:

$$\begin{aligned}2e(t)^TPB_{2i}\omega_1(t)&\le\psi_5e(t)^Te(t)+\psi_5^{-1}\|PB_{2i}\omega_1(t)\|^2\\&\le\psi_5e(t)^Te(t)+\psi_5^{-1}\delta_2^2\|PB_{2i}\|^2.\end{aligned}\tag{38}$$

In order to cancel the effect of the disturbances and uncertainties on the dynamics of the output system, $\varphi_i(t)$ and $\sigma_i(t)$ are selected as:

$$2e(t)^TP\varphi_i(t)=\psi_5^{-1}\delta_2^2\ \|\ PB_{2i}\ \|^2,\tag{39}$$

$$\begin{aligned}2e(t)^TP\sigma_i(t)=\big(&\psi_1(1+\psi_6)\zeta_{1i}^2\\&+\psi_3(1+\psi_7)\zeta_{2j}^2\big)\hat{x}(t)^T\hat{x}(t).\end{aligned}\tag{40}$$

Such that the performance criteria $\mathcal{J}(t)$ is:

$$\begin{aligned}\mathcal{J}(t)\le\sum_{i=1}^q\sum_{j=1}^q\hat{\mu}_i(\hat{\xi}(t))\hat{\mu}_j(\hat{\xi}(t))\Big(&e(t)^T\Gamma e(t)\\&+2e(t)^TPB_{3i}v(t)-\gamma^2v(t)^Tv(t)\Big)\end{aligned}\tag{41}$$

$$\begin{aligned}=\sum_{i=1}^q\sum_{j=1}^q\hat{\mu}_i(\hat{\xi}(t))\hat{\mu}_j(\hat{\xi}(t))\Bigg(&\begin{bmatrix}e(t)^T & v(t)^T\end{bmatrix}\\&\times\begin{bmatrix}\Gamma_{ij} & PB_{3i}\\ * & -\gamma^2I\end{bmatrix}\begin{bmatrix}e(t)\\ v(t)\end{bmatrix}\Bigg),\end{aligned}\tag{42}$$

where

$$\begin{aligned}\Gamma_{ij}=&(A_i-L_iC_j)^TP+P(A_i-L_iC_j)+\psi_1^{-1}P^2\\&+\psi_3^{-1}PL_i(PL_i)^T+\psi_2\zeta_{1i}^2I+\psi_4\zeta_{2j}^2I\\&+\psi_5I+G^TG.\end{aligned}$$

The analysis of these two cases prove that (30) holds if:

$$\begin{bmatrix}\Gamma_{ij} & PB_{3i}\\ * & -\gamma^2I\end{bmatrix}<0.\tag{43}$$

Due to the fact that inequality (43) is nonlinear, a change of variable $W_i=PL_i$ is performed in order to obtain a LMI representation. Finally, the Schur complement is considered to obtain the LMI given in Theorem 1, which can be easily solved with specialized software. This completes the proof. □

The condition in (30) is satisfied when the condition (28) holds. Therefore, both the stability and the $\mathcal{H}_\infty$ performance are achieved when the conditions (28) holds. Theorem 1 gives sufficient conditions, which guarantee the observer stability with $\mathcal{H}_\infty$ performance. Nonetheless, due to the fact that the time-varying matrix B_{3i} is uncertain as given in (17), with M representing the uncertain term, the condition given in (28) have an infinite dimension. Then, by considering Lemma 2, the matrix M is eliminated from Theorem 1, such that it can be rewritten as:

Theorem 2. Given the TS system (1) and the state observer (14), the estimation error (16) is asymptotically stable with $\mathcal{H}_\infty$ performance and attenuation level $\gamma>0$, if there exist a matrix $P=P^T>0$, matrices W_i and positive scalars $\psi_1,\psi_2,\psi_3,\psi_4,\psi_5$, and ψ_{8i} such that following optimization problem can be solved:

$$\min\quad\gamma$$

under the following LMI constraints:

$$\begin{bmatrix}\Lambda_{ij} & P\hat{B}_{1i} & P & W_i & P\Delta_{3i} & 0\\ * & -\gamma^2I & 0 & 0 & 0 & \psi_{8i}\tilde{B}_{1j}^T\\ * & * & -\psi_1I & 0 & 0 & 0\\ * & * & * & -\psi_3I & 0 & 0\\ * & * & * & * & -\psi_{8i}I & 0\\ * & * & * & * & * & -\psi_{8i}I\end{bmatrix}<0.\tag{44}$$

Then, the observer parameters are computed by:

$$L_i = P^{-1} W_i. \tag{45}$$

Proof. The conditions in (28) can be rewritten as,

$$\begin{bmatrix} \Lambda_{ij} & P\hat{B}_{1i} & P & W_i \\ * & -\gamma^2 I & 0 & 0 \\ * & * & -\psi_1 I & 0 \\ * & * & * & -\psi_3 I \end{bmatrix} + \begin{bmatrix} P\Delta_{3i} \\ 0 \\ 0 \\ 0 \end{bmatrix} M \begin{bmatrix} 0 & \tilde{B}_{1i} & 0 & 0 \end{bmatrix} + \left(\begin{bmatrix} P\Delta_{3i} \\ 0 \\ 0 \\ 0 \end{bmatrix} M \begin{bmatrix} 0 & \tilde{B}_{1i} & 0 & 0 \end{bmatrix} \right)^T < 0. \tag{46}$$

In terms of Lemma 2, the conditions (28) and (44) are equivalent. □

Remark 2. The main disadvantage of a TS system is that the number of sub-models increases significatively with respect to the number of nonlinear terms. In consequence, the computational load becomes heavy and, it can be difficult to find a feasible solution for Theorem 2. In order to avoid this problem, based on Lemma 3, new stability conditions are derived by relaxing the LMI conditions, such that the following Corollary is obtained:

Corollary 1. Given the TS system (1) and the state observer (14), the estimation error (16) is asymptotically stable with $\mathcal{H}_\infty$ performance and attenuation level $\gamma > 0$, if there exist a matrix $P = P^T > 0$, matrices W_i and positive scalars ψ_1, ψ_2, ψ_3, ψ_4, ψ_5 and ψ_{8i} such that the following optimization problem can be solved:

$$\min \quad \gamma$$

$$\Psi_{ii} < 0, i = 1, \ldots, q, \tag{47}$$

$$\frac{2}{q-1}\Psi_{ii} + \Psi_{ij} + \Psi_{ji} < 0, \quad i,j = 1, \ldots, q, i \neq j. \tag{48}$$

with:

$$\Psi_{ij} = \begin{bmatrix} \Lambda_{ij} & \begin{bmatrix} 0 & -W_i D \end{bmatrix} & P & W_i & P\Delta_{3i} & 0 \\ * & -\gamma^2 I & 0 & 0 & 0 & \psi_{8i}\tilde{B}_{1j}^T \\ * & * & -\psi_1 I & 0 & 0 & 0 \\ * & * & * & -\psi_3 I & 0 & 0 \\ * & * & * & * & -\psi_{8i} I & 0 \\ * & * & * & * & * & -\psi_{8i} I \end{bmatrix}.$$

Then, the observer parameters are computed by:

$$L_i = P^{-1} W_i. \tag{49}$$

Remark 3. It is important to note that the implementation of this observer induces a practical problem. This take place when the magnitude of $\varphi_i(t)$ and $\sigma_i(t)$ increases without limit due to the fact that the estimation error $e_y(t)$ tends to zero. This problem can be overcome by taking in consideration the following constraints:

If $e_y(t) \geq \epsilon$, then

$$\varphi_i = \delta_2^2 \psi_5^{-1} \frac{\|PB_{2i}\|^2}{2e_y(t)^T e_y(t)} P^{-1} \sum_{j=1}^{q} \hat{\mu}_j(\hat{\xi}) C_j^T e_y(t),$$

$$\sigma_i = \psi_{9i} \frac{\hat{x}^T \hat{x}}{2e_y(t)^T e_y(t)} P^{-1} \sum_{j=1}^{q} \hat{\mu}_j(\hat{\xi}) C_j^T e_y(t).$$

If $e_y < \epsilon$, then

$$\varphi_i = 0,$$
$$\sigma_i = 0,$$

where $\epsilon > 0$ is any small threshold parameter. With this new restriction, the estimation error cannot converge asymptotically to zero, but to a small neighborhood of zero depending on the magnitude of ϵ.

IV. FAULT DETECTION AND ISOLATION

Note that the previous development can be applied to detect and isolate faults by means of a bank of observers, which can adopt the well known generalized observer scheme (GOS) or the dedicated observer scheme (DOS) [34]. In the GOS, each observer is designed to be sensitive to all faults but one, while in the DOS, each observer is sensitive only to one fault. In order to exemplify the application of the proposed method, in this work a DOS is considered. In the DOS, separate observers are designed, in this way, the kth observer receives its scalar input $y^k(t)$ from only the kth sensor and each kth observer provides the estimated kth output. The plant input vector $u(t)$ is also applied to each of the kth dedicated observers.

If all the sensors are fault free, the estimated output $\hat{y}^k(t)$ will be identical to $y^k(t)$. Then, under sensor faults, system (1) can be represented as:

$$\dot{x}(t) = \sum_{i=1}^{q} \mu_i(\xi(t)) \left(A_i x(t) + B_{1i} u(t) + B_{2i}\omega(t) \right),$$
$$y(t) = \sum_{i=1}^{q} \mu_i(\xi(t)) C_i x(t) + Dv(t) + D_f f(t), \tag{50}$$

where $f(t)$ denotes the sensor faults vector, and $D_f = I_p$. Each observer takes the form:

$$\dot{\hat{x}}^k(t) = \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) \Big(A_i \hat{x}^k(t) + B_{1i} u + L_i^k (y^k(t) - \hat{y}^k(t)) + \varphi_i^k(t) + \theta_i^k(t) \Big), \tag{51}$$
$$\hat{y}^k(t) = \sum_{i=1}^{q} \hat{\mu}_i(\hat{\xi}(t)) C_i^k \hat{x}^k.$$

Then, the kth normalized residual is expressed as:

$$r_k(t) = \| y^k(t) - \hat{y}^k(t) \|, \tag{52}$$

such that, if $r_k(t)$ is less than a predefined threshold (th), then, the system is working under nominal conditions; but if $r_k > th$, then, the system is working on faulty conditions. For example, if the fault occurs in sensor 3, the residual $r_3(t)$ changes its value, but the other residues remains insensitive or close to zero. By considering all possible scenarios, a unique signature for each fault can be obtained as shown in Table I.

This scheme produces a decoupled sensor fault detection and isolation method. Nevertheless, it is not possible to detect simultaneous faults nor estimate the magnitude, which will be considered in future work.

Remark 4. In order to guarantee robustness against measurement noise, model mismatches, and the inexact premise variable, Corollary 1 have to be solved for each observer in the bank.

Table I. Incidence Matrix.

Fault	F_1	F_2	$\cdots$	F_k
$\|r_1(t)\|$	1	0	$\cdots$	0
$\|r_2(t)\|$	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\|r_k(t)\|$	0	0	$\cdots$	1

V. NUMERICAL EXAMPLES

This section includes two different examples to illustrate the effectiveness of the proposed approach. The first example considers a comparison with the case of unmeasurable premise variable, and the second example is given to illustrate the method in a more complex system.

5.1 Example 1

The aim of this numerical example is to compare the performance of the observer in H_∞ sliding mode presented in Section III against the case of unmeasurable premise variable. The observers are applied to a DC motor series in order to estimate the current $I(t)$ and the angular velocity $\omega(t)$. The TS model is presented in [35], which considers that $x_2(t) = I(t)$ is the premise variable and the output of the system. Note that, as stated in [35], $x_2(t)$ is measurable, however, in order to improve the observer performance, the authors consider $x_2(t)$ as unmeasurable. This example shows that in some cases, even if the premise variable is measurable, there may exist a certain degree of uncertainty, which reduces the applicability of traditional measurable approaches. The simulation is performed introducing measurement noise in the output and therefore in the premise variable. In order to minimize the effects of noise and uncertainty on the premise variable, Corollary 1 is solved and the following observer gains are obtained:

$$L_1 = \begin{bmatrix} -0.6703 \\ 11.4736 \end{bmatrix}, \; L_2 = \begin{bmatrix} -0.6704 \\ 11.4746 \end{bmatrix},$$
$$P = \begin{bmatrix} 1.5050 & 0.0879 \\ 0.0879 & 0.0720 \end{bmatrix}, \tag{53}$$

and $\psi_1 = 2.7060 \times 10^4$, $\psi_2 = 5.8403 \times 10^{-8}$, $\psi_3 = 8.2491 \times 10^4$, $\psi_4 = 1.7526 \times 10^{-5}$, $\psi_5 = 1.7525 \times 10^{-7}$.

The results of the state estimation are shown in Figs 1 and 2, for the angular velocity and the current, respectively. These results are compared with the results obtained in [35]. The observer of [35] is synthesized by using the differential mean value theorem and the nonlinear sector transformation. The estimation results are also displayed on Figs 1 and 2. It is clear that by considering the inexact approach, the observer performance and robustness are improved *i.e.* the convergence-time is reduced and the noises are well-attenuated. Nevertheless, it is also important to note that our approach can not be applied in the case of unmeasurable premise variables, due to the fact that noisy or inexact measurements are required.

5.2 Example 2

The lateral dynamics model of an electric vehicle is described by [30]:

$$\begin{aligned} \dot{v}_y(t) &= \frac{1}{m}\left(F_{yf}(t) + F_{yr}(t)\right) - v_x(t)\dot{\theta}(t), \\ \ddot{\theta}(t) &= \frac{1}{I_z}\left(a_f F_{yf}(t) + a_r F_{yr}(t)\right) + \frac{1}{I_z} M_z(t), \end{aligned} \tag{54}$$

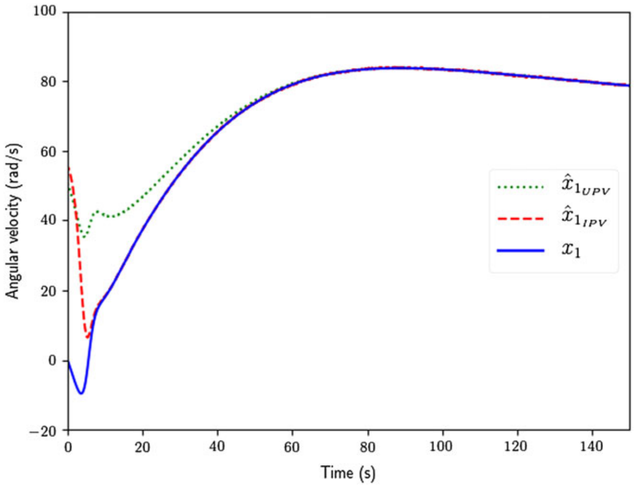

Fig. 1. Comparison between the TS observer with inexact premise variable (IPV) and the TS observer with unmeasurable premise variable (UPV) of [35]. [Color figure can be viewed at wileyonlinelibrary.com]

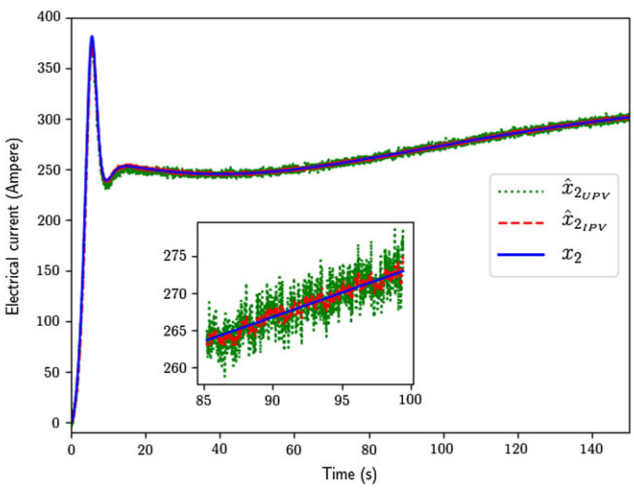

Fig. 2. Comparison between the TS observer with inexact premise variable (IPV) and the TS observer with unmeasurable premise variable (UPV) of [35]. [Color figure can be viewed at wileyonlinelibrary.com]

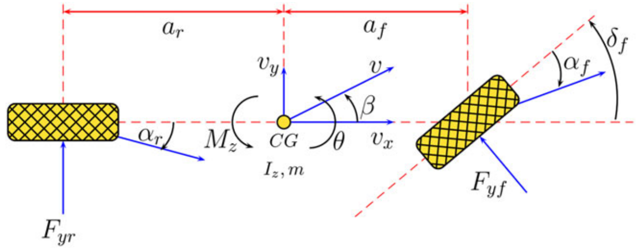

Fig. 3. Vehicle dynamics in yaw plane. [Color figure can be viewed at wileyonlinelibrary.com]

Table II. Parameters of the electric vehicle.

Parameter	value	unit
m	1500	kg
I_z	2454	kg m^2
η	0.1	-
a_f	1.1	m
a_r	1.44	m
Q_r	40000	-
Q_f	40000	-
v_{xmin}	5	m/s
v_{xmax}	30	m/s

where $v_y(t)$ and $\dot{\theta}(t)$ denote the lateral velocity and the yaw rate respectively. $F_{yf}(t)$ and $F_{yr}(t)$ are the lateral forces acting on the front and rear wheels respectively, displayed in Fig. 3. The parameters a_f and a_r represent the distance from the front and the rear wheel to the center of gravity. I_z is the yaw moment of inertia, m is the total mass of the vehicle, and $v_x(t)$ is the longitudinal velocity. In addition, $M_z(t)$ denotes the direct yaw moment on the yaw axis, which is induced due to the unequal tracking/braking forces on the tires. Due to the fact that the tracking/braking forces are difficult to measure or estimate (since the forces are related to the variable condition of the road), it is reasonable to assume that the direct yaw moment $M_z(t)$ is unknown [17]. The parameter values of the model considered in this example are shown in Table II.

Because of the characteristics of the tires, it is generally assumed that forces $F_{yf}(t)$ and $F_{yr}(t)$ are generated by a dynamical system described by:

$$\begin{aligned} \frac{\eta_f}{v_x}\dot{F}_{yf}(t) + F_{yf}(t) &= F^S_{yf}(t), \\ \frac{\eta_r}{v_x}\dot{F}_{yr}(t) + F_{yr}(t) &= F^S_{yr}(t), \end{aligned} \tag{55}$$

which takes into account the transient phase of the tires response.η_i, $i \in \{r,f\}$ are the relaxation lengths, which are positive scalars. The inputs $F^S_{yf}(t)$ and $F^S_{yr}(t)$ are the steady-state (static) forces expressed by the 'magic formula' of Pacejka [36]:

$$F^S_{yi}(t) = \mathcal{D}_i \sin\left(\mathcal{C}_i \tan^{-1}\left(\mathcal{B}_i(1-\mathcal{E}_i)\alpha_i(t) + \mathcal{E}_i tan^{-1}\left(\mathcal{B}_i\alpha_i(t)\right)\right)\right),\ i \in \{r,f\}, \tag{56}$$

where $\mathcal{B}_i$, $\mathcal{C}_i$, $\mathcal{D}_i$ and $\mathcal{E}_i$ are parameters depending on the characteristic of the tires and the road, $\alpha_f(t)$ and $\alpha_r(t)$ represent the tire slip angles of the front and the rear wheels, respectively, which are expressed as:

$$\begin{aligned}\alpha_f(t) &= -\frac{v_y(t)}{v_x(t)} - \tan^{-1}\left(\frac{a_f}{v_x(t)}\dot{\theta}(t)\cos\left(\frac{v_y(t)}{v_x(t)}\right)\right) + \delta_f(t),\\ \alpha_r(t) &= -\frac{v_y(t)}{v_x(t)} - \tan^{-1}\left(\frac{a_r}{v_x(t)}\dot{\theta}(t)\cos\left(\frac{v_y(t)}{v_x(t)}\right)\right),\end{aligned} \tag{57}$$

where $\delta_f(t)$ is the front steering angle. The body side slip angle is defined by $\beta(t) = \tan^{-1}\left(\frac{v_y(t)}{v_x(t)}\right)$. In normal driving situations, the lateral velocity is small, which allows to approximate the sideslip angle by $\beta(t) \approx \frac{v_y(t)}{v_x(t)}$; this angle is also small in driving mode. If, the wheel sideslip angles $\alpha_f(t)$ and $\alpha_r(t)$ are not greater than 8 degrees, then, (57) can be simplified as follows:

$$\begin{aligned}\alpha_f(t) &= -\frac{v_y(t)}{v_x(t)} - \frac{a_f}{v_x(t)}\dot{\theta}(t) + \delta_f(t),\\ \alpha_r(t) &= -\frac{v_y(t)}{v_x(t)} - \frac{a_r}{v_x(t)}\dot{\theta}(t).\end{aligned} \tag{58}$$

Also, if the forces $F^S_{yf}(t)$ and $F^S_{yr}(t)$ are in the linear zone [30], then, they can be expressed by the linear expressions:

$$\begin{aligned}F^S_{yf}(t) &= Q_f\left(-\frac{v_y(t)}{v_x(t)} - \frac{a_f}{v_x(t)}\dot{\theta}(t) + \delta_f(t)\right),\\ F^S_{yr}(t) &= Q_r\left(-\frac{v_y(t)}{v_x(t)} - \frac{a_r}{v_x(t)}\dot{\theta}(t)\right),\end{aligned} \tag{59}$$

where $Q_f = \mathcal{D}_f\mathcal{C}_f\mathcal{B}_f$ and $Q_r = \mathcal{D}_r\mathcal{C}_r\mathcal{B}_r$.

Finally, by using the following change of variables:

$$\begin{aligned}x_1(t) &= v_y(t),\\ x_2(t) &= \dot{\theta}(t),\\ x_3(t) &= \frac{1}{m}\left(F_{yf}(t) + F_{yr}(t)\right),\\ x_4(t) &= \frac{1}{I_z}\left(a_f F_{yf}(t) - a_r F_{yr}(t)\right),\end{aligned} \tag{60}$$

the system becomes:

$$\begin{aligned}\dot{x}_1 &= -v_x(t)x_2(t) + x_3(t),\\ \dot{x}_2 &= x_4(t) + \frac{1}{I_z}M_z(t),\\ \dot{x}_3 &= -\frac{v_x(t)}{\eta}x_3(t) + \frac{v_x(t)}{m\eta}\left(F^S_{yf}(t) + F^S_{yr}(t)\right),\\ \dot{x}_4 &= -\frac{v_x(t)}{\eta}x_4(t) + \frac{v_x(t)}{I_z\eta}\left(a_f F^S_{yf}(t) - a_r F^S_{yr}(t)\right).\end{aligned} \tag{61}$$

This change of variables aims at scaling the state variables and the matrices in order to reduce the conservatism related to the LMI constraints. Note also that the relaxation terms η_f and η_r are considered identical and denoted by η. By assuming that the longitudinal velocity is time-varying, which is more realistic than a constant one (as commonly used in the literature), and by expressing the system in matrix formulation, System (61) can be written as:

$$\dot{x}(t) = A\left(v_x(t)\right)x(t) + B_1\left(v_x(t)\right)u(t) + B_2\omega(t), \tag{62}$$

with $\omega(t) = M_z(t)$ is an unknown input and $u(t) = \delta_f(t)$ is known but not controllable and where

$$A(v_x(t)) = \begin{pmatrix} 0 & -v_x(t) & 1 & 0\\ 0 & 0 & 0 & 1\\ a_{31} & a_{32} & a_{33}v_x(t) & 0\\ a_{41} & a_{42} & 0 & a_{44}v_x(t)\end{pmatrix},$$

$$B_1(v_x(t)) = \begin{pmatrix} 0\\ 0\\ b_3 v_x(t)\\ b_4 v_x(t)\end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0\\ \frac{1}{I_z}\\ 0\\ 0\end{pmatrix},$$

and

$$a_{31} = -\frac{Q_f}{m\eta} - \frac{Q_r}{m\eta}, \quad a_{32} = \frac{Q_r a_r}{m\eta} - \frac{Q_f a_f}{m\eta},$$
$$a_{33} = a_{44} = -\frac{1}{\eta}, \quad a_{41} = \frac{a_r Q_r}{I_z\eta} - \frac{a_f Q_f}{I_z\eta},$$
$$a_{42} = -\frac{a_f^2 Q_f}{I_z\eta} - \frac{a_r^2 Q_r}{I_z\eta}, \quad b_3 = \frac{Q_f}{m\eta}, b_4 = \frac{a_f Q_f}{I_z\eta}.$$

By considering the longitudinal velocity as the premise variable, which is bounded as $\underline{\xi} \le \xi(t) = v_x(t) \le \overline{\xi}$ and using the sector-nonlinearity approach [19], the following TS model is obtained:

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(\xi(t)) \left(A_i x(t) + B_{1_i} u(t) + B_2 w(t) \right), \quad (63)$$

where the activating functions are defined by

$$\mu_1(\xi(t)) = \frac{\xi(t) - \underline{\xi}}{\overline{\xi} - \underline{\xi}},$$
$$\mu_2(\xi(t)) = \frac{\overline{\xi} - \xi(t)}{\overline{\xi} - \underline{\xi}},$$

and the sub-model matrices are given by

$$A_1 = \begin{pmatrix} 0 & -\overline{\xi} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33}\overline{\xi} & 0 \\ a_{41} & a_{42} & 0 & a_{44}\overline{\xi} \end{pmatrix},$$
$$B_{1\delta_f} = \begin{pmatrix} 0 \\ 0 \\ b_3\overline{\xi} \\ b_4\overline{\xi} \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & -\underline{\xi} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33}\underline{\xi} & 0 \\ a_{41} & a_{42} & 0 & a_{44}\underline{\xi} \end{pmatrix},$$
$$B_{1\delta_f} = \begin{pmatrix} 0 \\ 0 \\ b_3\underline{\xi} \\ b_4\underline{\xi} \end{pmatrix}.$$

Given that the nonlinear sector transformation [19] was considered to obtain the TS model, it is assumed

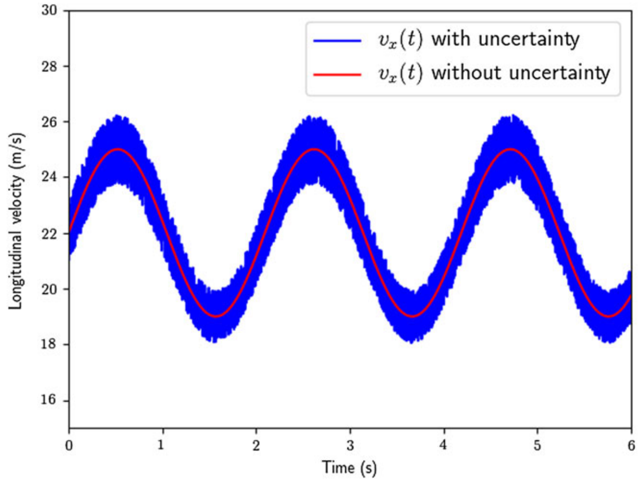

Fig. 4. Longitudinal velocity of vehicle. [Color figure can be viewed at wileyonlinelibrary.com]

that the nonlinear model (62) is exactly represented by (63), without loss of information, in an operation region, which is bounded by the maximum and minimum values of the premise variables.

In this work, it is considered that the vehicle is equipped with sensors that measure the yaw rate $\dot{\theta}(t)$ and the lateral acceleration $a_y(t)$. Due to the fact that $a_y(t) = \frac{1}{m}\left(F_{yf} + F_{yr}\right) - v_x(t)\dot{\theta}(t) = x_3(t) - \xi(t)x_2(t)$ and taking into account additive sensor faults $f(t)$, which can affect the sensors, the output equation is defined as follows:

$$y(t) = \sum_{i=1}^{2} \mu_i(\xi(t)) C_i x(t) + D_f f(t), \quad (64)$$

where

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\overline{\xi} & 1 & 0 \end{pmatrix},$$
$$C_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\underline{\xi} & 1 & 0 \end{pmatrix}.$$

In order to detect and isolate faults in the sensors, a bank of two observers were designed under the DOS described in (51). To design the first observer the output matrices are

$$C_1^1 = C_2^1 = [\,0 \ 1 \ 0 \ 0\,], \quad (65)$$

and for the second observer:

$$C_1^2 = [\,0 \ -\overline{\xi} \ 1 \ 0\,], \quad (66)$$

$$C_2^2 = [\,0 \ -\underline{\xi} \ 1 \ 0\,]. \quad (67)$$

For both observers, the value of the matrix G is selected as $G = [0\ 1\ 0\ 0]$.

It is necessary to mention that, in electric vehicles, the longitudinal velocity $v_x(t)$ is directly estimated by considering the rotational velocity of the motor. However, due to the slipping of the vehicle in the longitudinal direction, the longitudinal velocity cannot be estimated with precision. This effect, implies that in the TS model, the weighting functions $\mu_1(\xi(t))$ and $\mu_2(\xi(t))$ are also inexact. To overcome this problem, it is considered that the weighting functions have certain degree of uncertainty, which is delimited as follows:

$$\rho_i \in [\,0.9\ \ 1.1\,],\ \forall i = 1, 2.$$

In consequence, it is possible to compute $\|\Delta A_i\|$ and $\|\Delta C_i^k\|$ from (10) and (11); and the upper bound of the unknown input, $M_z(t)$, is the maximal longitudinal force multiplied by one half of vehicle width [17]. Then, by considering previous remarks, the observer gains are calculated by solving Corollary 1 with the Yalmip Toolbox [37]. For the first observer, the gain matrices are given by:

$$L_1^1 = \begin{bmatrix} 0.0109 \\ -0.0231 \\ 0.0436 \\ 1.1034 \end{bmatrix},\ L_2^1 = \begin{bmatrix} -0.0185 \\ -0.0585 \\ -0.1261 \\ -4.1262 \end{bmatrix},$$

$$P^1 = \begin{bmatrix} 0.0093 & -0.0001 & -0.0010 & 0.0001 \\ -0.0001 & 0.0096 & 0.0002 & -0.0010 \\ -0.0010 & 0.0002 & 0.0081 & 0.0000 \\ 0.0001 & -0.0010 & 0.0000 & 0.0081 \end{bmatrix},$$

and

$$\psi_1^1 = 0.4835,\ \psi_2^1 = 0.0080,\ \psi_3^1 = 0.5637,\ \psi_4^1 = 1.1274,\ \psi_5^1 = 1.2165.$$

For the second observer, the gain matrices are:

$$L_1^2 = \begin{bmatrix} -0.1117 \\ 0.0978 \\ 0.0694 \\ -3.0835 \end{bmatrix},\ L_2^2 = \begin{bmatrix} 0.2048 \\ -0.1109 \\ -0.2602 \\ 14.5403 \end{bmatrix},$$

$$P^2 = \begin{bmatrix} 0.0063 & -0.0002 & -0.0009 & 0.0001 \\ -0.0002 & 0.0067 & 0.0002 & -0.0009 \\ -0.0009 & 0.0002 & 0.0050 & -0.0000 \\ 0.0001 & -0.0009 & -0.0000 & 0.0051 \end{bmatrix};$$

and

$$\psi_1^2 = 0.3273,\ \psi_2^2 = 5.6837 \times 10^{-4},\ \psi_3^2 = 0.3820,\ \psi_4^2 = 1.7662,\ \psi_5^2 = 1.0187.$$

To demonstrate the applicability of the method, numerical simulations are done by considering white

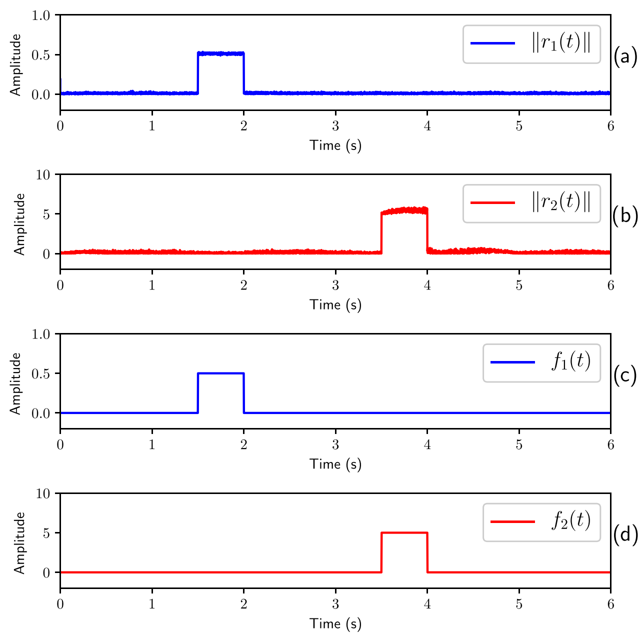

Fig. 5. Normalized residuals vectors (a) $\|r_1(t)\|$, (b) $\|r_2(t)\|$; and induced fault signals (c) $f_1(t)$, (d) $f_2(t)$. [Color figure can be viewed at wileyonlinelibrary.com]

noise with noise power of 1×10^{-6} in the signals of the sensors applied to the DOS scheme. Sinusoidal signals were considered as the known and unknown input, that are $u(t) = 0.05\sin(2t)$ and $w(t) = 50\sin(10t) + 100$ for all t. The simulation is done by considering an initial condition of $x(0) = \hat{x}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$.

Fig. 4 shows the premise variable v_x in the test, the red curve shows the premise variable without uncertainty and the blue curve illustrates the simulated uncertain measurements of the premise variable and the uncertain factor is performed with the Simulink block Uniform Random Number (minimum 0.95 and maximum 1.05).

To demonstrate the applicability of the FDI scheme, a fault is induced to the yaw rate sensor (sensor 1), denoted as f_1, and a fault is induced to the lateral acceleration sensor (sensor 2), denoted as f_2, both faults are shown in Fig. 5c,d. Note that sensor faults can be observed as additive bias, for example, offsets or calibration problems. These malfunctions can be described by ramp or step functions in order to represent abrupt or slow variation faults. In this work, abrupt faults are considered. The normalized residual signals are shown in Fig. 5. In fault-free operation, the observers can estimate the real states with a small error despite the sensor noise and the uncertainty given in the premise variables. However, at the time of the first fault, the first residual changes abruptly, which is used to detect and isolate the fault in sensor 1. In both cases, fault detection is successful. The determination of the fault is made by comparing the signature of the fault with the incident matrix given in Table I. The proposed fault diagnosis scheme is effective to detect faults in both sensors even in the presence of uncertainty in the weighting functions.

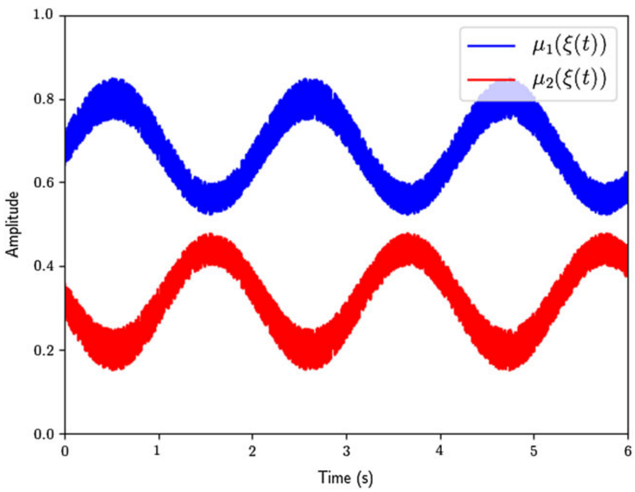

Fig. 6. Uncertain weighting functions. [Color figure can be viewed at wileyonlinelibrary.com]

Finally, in Fig. 6, the behavior of the weighting functions affected by the uncertainty in the premise variables are shown. It can be noted that the region of operation changes periodically due to the variation in the premise variable. In this work, the measurements of the premises variables are fault free.

VI. CONCLUSION

In this work, a $\mathcal{H}_\infty$ sliding mode and unknown input observer for a Takagi-Sugeno system was proposed. It was considered that the TS system was affected by premise variables that are measured with a certain degree of uncertainty. It represents a difficult problem concerning the computation of the observer gains, but it increases the applicability of the method. The strategy used was based on the $\mathcal{H}_\infty$ performance criteria to be robust against disturbances, sensor noise and uncertainty induced by inexact premise variables. As a result, the observer gains were obtained by the solution of a set of relaxed LMI conditions. Furthermore, it was demonstrated that the proposed approach is suitable to detect and isolate sensor faults by means of a dedicated observer scheme. Finally, a numerical example of the lateral dynamics of an electric vehicle model was presented to show the effectiveness and applicability of the proposed approach.

Future work will be done to extend the method to actuator fault diagnosis and fault tolerant control. Note that, a reconfigurable controller can be designed in order to maintain stability, acceptable dynamic performance and steady state of the system, in the event of a fault, based on the proposed fault detection method.

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